RESEARCH

A very elementary proof of an explicit formula of Bernoulli Numbers

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ABSTRACT

The aim of this paper is to give an easy and very elementary proof of a well-known explicit formula for Bernoulli numbers.

Key words: *Stirling numbers of the second kind; Bernoulli numbers; Bernoulli polynomials*

INTRODUCTION

The numbers :

$b_0 = 1,\quad b_2 = \frac{1}{6},\quad b_4 = -\frac{1}{30},\quad b_6 = \frac{1}{42},$

$b_8 = -\frac{1}{30}\quad \ldots,\quad b_1 = \frac{1}{2},\quad b_3 = b_5 = b_7 = b_9 = \cdots = 0$

are called the Bernoulli numbers. They can be defined by the following exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$$

When this sequence of numbers appeared in 1713, mathematicians didn't know a formula to compute b_n directly, so they used recursive formulas like this one [1]:

$$\begin{cases} b_0 = 1 \\ \forall n \geq 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 \end{cases}$$

In 1883, Worpitzky gave the following formula for b_n [2]:

$$b_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{i=0}^{k} \binom{k}{i} (-1)^i i^n$$

(1)

We can also find other mathematicians from the 19th century who proved formula (1), such as Cesàro in 1885 and d'Ocagne in 1889 [3, 4].

For our part, we present here an elementary proof of the formula (1).

STIRLING NUMBERS OF THE SECOND KIND

Let Y be an arbitrary function and set:

$$\begin{aligned} D^1 Y &= x\frac{d}{dx} Y \\ D^2 Y &= x\frac{d}{dx} D^1 Y \\ D^3 Y &= x\frac{d}{dx} D^2 Y \\ &\ldots \\ D^n Y &= x\frac{d}{dx} D^{n-1} Y \end{aligned}$$

If we develop the first four functions, we find:

$D^1 Y = xY'$

$D^2 Y = xY' + x^2 Y''$

$D^3 Y = xY' + 3x^2 Y'' + x^3 Y^{(3)}$

$D^4 Y = xY' + 7x^2 Y'' + 6x^3 Y^{(3)} + x^4 Y^{(4)}$

...

We conjecture that:

$$D^n Y = S_n^0 Y + S_n^1 x Y' + S_n^2 x^2 Y'' + \ldots + S_n^n x^n Y^{(n)} \qquad (2)$$

The coefficients S_n^k are called Stirling numbers of the second kind. They can be represented in a triangle similar to Pascal's triangle. The triangle of the numbers S_n^k is the following:

TABLE 1
The law for forming the numbers

	k=0	k=1	k=2	k=3	k=4	…
n=0	1					
n=1	0	1				
n=2	0	1	1			
n=3	0	1	3	1		
n=4	0	1	7	6	1	
…	…	…	…	…	…	…

We observe that:

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$$\begin{cases} S_0^0 = 1 \\ \forall n \geq 1, \quad S_n^0 = 0 \end{cases}$$

The law for forming the numbers in the above table is given by:

$$S_n^k = S_{n-1}^{k-1} + kS_{n-1}^k$$

THE EXPLICIT FORMULA FOR STIRLING NUMBERS OF THE SECOND KIND

If we put $Y = e^x$ in the formula (2), we obtain:

$$D^n e^x = e^x \sum_{k=0}^{n} S_n^k x^k$$

$$\Rightarrow \quad e^{-x}.D^n e^x = \sum_{k=0}^{n} S_n^k x^k$$

$$\Rightarrow \quad \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right).D^n \left(\sum_{i=0}^{\infty} \frac{x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k x^k$$

$$\Rightarrow \quad \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{D^n x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k x^k$$

One can easily prove that $D^n x^i = i^n x^i$, so:

$$\left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{i^n x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k \, x^k$$

If we develop the left-hand side we obtain:

$$\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} \frac{(-1)^{k-i}\binom{k}{i} i^n}{k!}\right) x^k = \sum_{k=0}^{n} S_n^k \, x^k$$

Comparing coefficients in both summations, we conclude that:

$$S_n^k = \frac{1}{k!}\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \tag{3}$$

RELATION BETWEEN BERNOULLI NUMBERS AND STIRLING NUMBERS OF THE SECOND KIND

Putting $Y = x^y$ in the formula (2), we get:

$$D^n x^y = \sum_{k=0}^{n} S_n^k \, x^k (x^y)^{(k)}$$

We know that $(x^y)^{(k)} = y(y-1)\dots(y-k+1)x^{y-k}$ and $D^n x^y = y^n x^y$so we get:

$$y^n = \sum_{k=0}^{n} S_n^k y(y-1)\dots(y-k+1) \tag{4}$$

The polynomial $y(y-1)\dots(y-k+1)$ is called the falling factorial of y of order k. Pochhammer used the symbol $(y)_k$ to denote it, so the formula (4) becomes using Pochhammer symbol:

$$y^n = \sum_{k=0}^{n} S_n^k (y)_k \tag{4'}$$

One interesting property of the falling factorial function is the following:

Proposition
Let y, n be non-negative integers, then:

$$(y+1)_{n+1} - (y)_{n+1} = (n+1)(y)_n$$

Proof

$$\begin{aligned} (y+1)_{n+1} - (y)_{n+1} &= (y+1)y(y-1)\dots(y-n+1) - y(y-1)\dots(y-n+1)(y-n) \\ &= \left[(y+1)-(y-n)\right]y(y-1)\dots(y-n+1) \\ &= (n+1)(y)_n \end{aligned}$$

We are going to use this property in the proof of the following proposition.

Proposition
Let $(n,m) \in \mathbb{N}^2$. We have:

$$\sum_{y=0}^{m} y^n = \sum_{k=0}^{n} S_n^k \frac{(m+1)_{k+1}}{k+1} \tag{5}$$

Proof
If we sum for y in the formula (4') we find:

$$\sum_{y=0}^{m} y^n = \sum_{y=0}^{m}\left(\sum_{k=0}^{n} S_n^k \, (y)_k\right)$$

$$\Rightarrow \quad \sum_{y=0}^{m} y^n = \sum_{k=0}^{n} S_n^k \left(\sum_{y=0}^{m} (y)_k\right)$$

$$\Rightarrow \quad \sum_{y=0}^{m} y^n = \sum_{k=0}^{n} S_n^k \left(\sum_{y=0}^{m} \frac{(y+1)_{k+1} - (y)_{k+1}}{k+1}\right)$$

$$\Rightarrow \quad \sum_{y=0}^{m} y^n = \sum_{k=0}^{n} S_n^k \left(\frac{(m+1)_{k+1} - (0)_{k+1}}{k+1}\right)$$

Therefore:

$$\sum_{y=0}^{m} y^n = \sum_{k=0}^{n} S_n^k \frac{(m+1)_{k+1}}{k+1}$$

Definition
Let $\in \mathbb{N}$
The Bernoulli polynomials $B_n(x)$ are defined by the following generating function:

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

One interesting observation to make about Bernoulli polynomials is that if we put $x = 0$ we get:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(0) \frac{t^n}{n!}$$

This generating function corresponds to the generating function of Bernoulli numbers b_n. Hence for all $n \in \mathbb{N}$, we have:

$B_n(0) = b_n$

Another interesting property of the Bernoulli polynomials is the following:

Proposition
Let $n \in \mathbb{N}$

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

Proof
On the one hand:

$$\begin{aligned}\sum_{n=0}^{\infty}\{B_n(x+1) - B_n(x)\}\frac{t^n}{n!} &= \frac{te^{t(x+1)}}{e^t-1} - \frac{te^{tx}}{e^t-1} \\ &= \frac{te^{tx}.e^t - te^{tx}}{e^t-1} \\ &= \frac{te^{tx}(e^t-1)}{e^t-1} \\ &= te^{tx}\end{aligned}$$

On the other hand:

$$\begin{aligned}\sum_{n=0}^{\infty} nx^{n-1}\frac{t^n}{n!} &= \sum_{n=1}^{\infty} t\frac{(xt)^{n-1}}{(n-1)!} \\ &= te^{xt}\end{aligned}$$

Comparing coefficients of both summations we conclude that:

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

Proposition
Let $n \in \mathbb{N}$

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k} b_{n-k}x^k$$

Proof

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} &= \frac{te^{tx}}{e^t-1} \\ &= \frac{t}{e^t-1}.e^{tx} \\ &= \left(\sum_{n=0}^{\infty} b_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k}\frac{t^{n-k}}{(n-k)!}.\frac{(xt)^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k}\binom{n}{k}x^k\right)\frac{t^n}{n!}\end{aligned}$$

Therefore:

$$B_n(x) = \sum_{k=0}^{n} b_{n-k}\binom{n}{k}x^k$$

If we sum for y in the relation $B_{n+1}(y+1) - B_{n+1}(y) = (n+1)y^n$, we obtain :

$$\begin{aligned}(n+1)\sum_{y=0}^{m} y^n &= \sum_{k=0}^{m}\{B_{n+1}(y+1) - B_{n+1}(y)\} \\ &= B_{n+1}(m+1) - B_{n+1}(0) \\ &= B_{n+1}(m+1) - b_{n+1}\end{aligned}$$

Thus:

$$(n+1)\sum_{y=0}^{m} y^n = B_{n+1}(m+1) - b_{n+1} \qquad (6)$$

Comparing formula (5) with formula (6). We conclude that:

$$B_{n+1}(m+1) - b_{n+1} = (n+1)\sum_{k=0}^{n} S_n^k\frac{(m+1)_{k+1}}{k+1} \qquad (7)$$

If we develop the expression of $(X)_{k+1}$ in terms of the powers of X we find:

$$\begin{aligned}(X)_{k+1} &= X(X-1)\dots(X-k) \\ &= X\left(X^k - \frac{k(k+1)}{2}X^{k-1} + \dots + (-1)^k k!\right) \\ &= X\sum_{j=0}^{k} c_j X^j \\ &= \sum_{j=0}^{k} c_j X^{j+1}\end{aligned}$$

Therefore:

$$(X)_{k+1} = \sum_{j=0}^{k} c_j X^{j+1}$$

If we apply the above formula for $(m+1)_{k+1}$ in the formula (7) we find:

$$B_{n+1}(m+1) - b_{n+1} = \sum_{k=0}^{n} S_n^k\frac{n+1}{k+1}\sum_{j=0}^{k} c_j (m+1)^{j+1}$$

Substituting also $B_{n+1}(m+1)$ by its explicit expression, we finally get:

$$\begin{aligned}&\left(\sum_{k=0}^{n+1}\binom{n+1}{k} b_{n+1-k}(m+1)^k\right) - b_{n+1} = \sum_{k=0}^{n} S_n^k\frac{n+1}{k+1}\sum_{j=0}^{k} c_j (m+1)^{j+1} \\ \Rightarrow &\sum_{k=1}^{n+1}\binom{n+1}{k} b_{n+1-k}(m+1)^k = \sum_{k=0}^{n} S_n^k\frac{n+1}{k+1}\sum_{j=0}^{k} c_j (m+1)^{j+1} \\ \Rightarrow &\sum_{j=0}^{n}\binom{n+1}{j+1} b_{n-j}(m+1)^{j+1} = \sum_{k=0}^{n} S_n^k\frac{n+1}{k+1}\sum_{j=0}^{k} c_j (m+1)^{j+1} \\ \Rightarrow &\sum_{j=0}^{n}\left(\binom{n+1}{j+1} b_{n-j}\right)(m+1)^j = \sum_{j=0}^{n}\left(\sum_{k=j}^{n} S_n^k\frac{n+1}{k+1}c_j\right)(m+1)^j\end{aligned}$$

We have equality between two polynomials in $m+1$, both of degree n, so the coefficients of the terms of the same degree are equal. In particular for $j=0$ we have:

$$\binom{n+1}{1} b_n = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} c_0$$

$$\Longrightarrow \qquad b_n = \sum_{k=0}^{n} S_n^k \frac{(-1)^k k!}{k+1}$$

To get the explicit expression of b_n in terms of n, we substitute S_n^k in the above identity by its explicit expression, and after simplification we obtain the remarkable formula (1) for the Bernoulli numbers.

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