EDITORIAL

Dominating sets and domination polynomials of cubic paths

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Medona A, Christilda S. Dominating sets and domination polynomials of cubic paths. J Pure Appl Math 2022;6(5): 32-35

ABSTRACT

Let G = (V, E) be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in V – S is adjacent to at least one vertex in S. Let P_n^3 be the cubic path P_n and let $D(P_n^3, i)$ denote the family of all dominating sets of P_n^3 with cardinality i. Let $d(P_n^3, i) = |D(P_n^3, i)|$. In this paper, we obtain a recursive formula for $d(P_n^3, i)$. Using this recursive formula, we construct the polynomial $D(P_n^3, i)x^i = \sum_{i=\lceil \frac{n}{7} \rceil}^{n} d(P_n^3, i)x^i$ which we call the domination polynomial of Pn3 and obtain some properties of this polynomial.

Keywords: *Domination Set; Domination Number; Domination Polynomials*

INTRODUCTION

Mets Let $G = (V, E)$ be a simple graph of order $|v| = n$. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of G, if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to atleast one vertex in S. The domination number of a graph G is defined as the minimum size of a dominating set of vertices in G and it is denoted by $\gamma(G)$. A simple path is a path in which all its internal vertices have degree two and the end vertices have degree one and is denoted by k^{th}.

Definition 1

The k^{th} power of a graph is a graph with set of vertices of G and an edge between two vertices if and only if there is a path of length atmost k between them. It is denoted by G_n^k and also called k^{th} power of G.

Definition 2

Let $D(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |D(G, i)|$ then the domination polynomial $D(G, x)$ of G is defined by $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$

Where $\gamma(G)$ is the domination number of G.

Definition 3

The cube of a graph with the same set of vertices as G and an edge between two vertices and only if there is path of length atmost 3 between them. The third power of a graph is also called its cube of G [1, 2].

Let P_n^3 be the cubic of the path P_n (3rd power) with n vertices. Let $D(P_n^3, i)$ be the family of dominating sets of the graph with cardinality i and let $d(P_n^3, i) = |D(p_n^3, i)|$ we call the polynomial $D(P_n^3, x) = \sum_{i=\lceil \frac{n}{7} \rceil}^{n} d(P_n^3, i)x^i$

MAIN RESULT

Let $D(P_n^3, i)$ be the family of dominating sets of P_n^3 with cardinality i. we investigate the dominating sets of P_n^3, we need the following lemma to prove our main results in this section [3].

Lemma 1

$$\gamma(P_n^3) = \left\lceil \frac{n}{7} \right\rceil$$

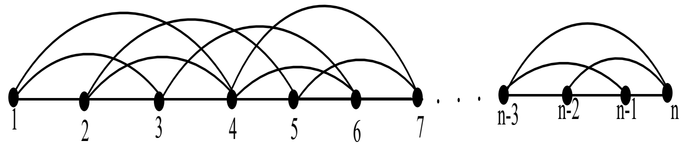

Figure 1) *Proof* P_n^3

Proof

In the proof P_n^3, any vertex i with $4 \le i \le n - 3$ covers i – 1 and i – 3 in the left side and i + 1 and i + 3 in the right side. Similarly any vertex i with $3 \le i \le n - 2$ covers i – 1 and i – 2 in the left side and i+1 and i+2 in the right side. Therefore, a single vertex covers atmost 7 vertices Figure 1.

Therefore $\gamma(P_n^3) = \left\lceil \frac{n}{7} \right\rceil$

Domination Polynomial of P_n^3

Let $D(P_n^3, x) = \sum_{i=\lceil \frac{n}{7} \rceil}^{n} d(P_n^3, i)x^i$ be the domination polynomial of a cubic path P_n^3. In this section we derive the expression for $D(P_n^3, \chi)$.

Example 1

The graph P_4^3 has one dominating set of cardinality 4, 4 dominating set of cardinality 3, 6 dominating set of cardinality 2, 4 dominating set of cardinality 1.

Therefore its domination polynomial is $D(P_4^3, x) = x^4 + 4x^3 + 6x^2 + 4x$.

RESULT

If $D(P_n^3)$ is the family of dominating sets with cardinality i of P_n^3, then $d(P_n^3, i) = d(P_{n-1}^3, i-1) + d(P_{n-2}^3, i-1) + d(P_{n-3}^3, i-1) + d(P_{n-4}^3, i-1) + d(P_{n-5}^3, i-1) + d(P_{n-6}^3, i-1) + d(P_{n-7}^3, i-1)$

Where $d(P_n^3, i) = |D(P_n^3, i)|$

We obtain $d(P_n^3, i)$ for $1 \le n \le 15$ as shown in following table $d(P_n^3, i)$ the number of dominating set of P_n^3 with cardinality i

In the following theorem we obtain some properties of d (Pn3, i) Figure 2, Table 1.

Theorem 1

The following properties hold for the coefficient of $D(P_4^3, x)$.

i) $d(P_{7n}^3, n) = 1$ for every $n \in N$

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Received: 21-Sep-2022, Manuscript No. PULJPAM-22-5374, Editor Assigned: 23-Sep-2022, PreQC No. PULJPAM-22-22-5374 (PQ), Reviewed: 30-Sep-2022, QC No. puljpam-22-5374 (Q), Revised: 02-Oct-2022, Manuscript No PULJPAM-22-5374 (R), Published: 20-Oct-2022, DOI:-10.37532/2752-8081.22.6(5).32-35.

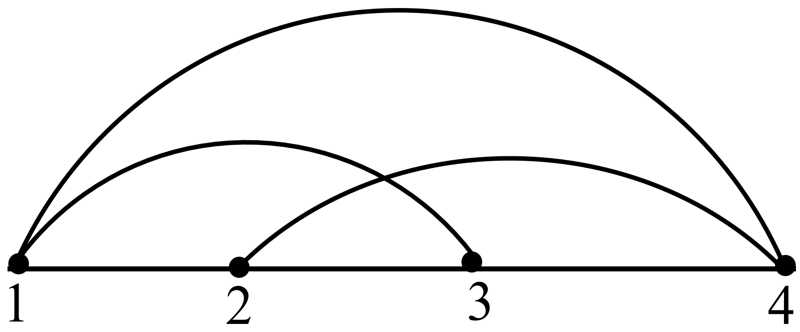

Figure 2) *Graph* P_4^3.

TABLE 1

In the following theorem we obtain some properties of $d\left(P_n^3,\ i\right)$

in	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1														
2	2	1													
3	3	3	1												
4	4	6	4	1											
5	3	10	10	5	1										
6	2	13	20	15	6	1									
7	1	15	33	35	21	7	1								
8	0	16	48	68	56	28	8	1							
9	0	15	64	116	124	84	36	9	1						
10	0	13	78	180	240	208	120	45	10	1					
11	0	10	88	257	420	448	328	165	55	11	1				
12	0	6	92	341	676	868	776	433	220	66	12	1			
13	0	3	88	423	1012	1543	1644	1269	653	286	78	13	1		
14	0	1	78	491	1420	2549	3186	2913	1922	939	364	91	14	1	
15	0	0	64	536	1876	3948	5728	6098	4835	2861	1303	455	105	15	1

ii) $d\left(P_n^3,\ n\right)\ =\ 1$ for every $n \in N$

iii) $d(P_n^3,\ n-1)\ =\ n$ for every $n \geq 2$

iv) $d(P_{7n}^3,\ n-2)\ =\ nC_2$ for every $n \geq 3$

v) $d(P_n^3,\ n-3)\ =\ nC_3$ for every $n \geq 4$

vi) $d(P_n^3,\ n-4)\ =\ nC_4 -\ 2$ for every $n \geq 5$

vii) $d(P_n^3,\ n-5)\ =\ nC_5{}^{-}2\left(n\ -\ 4\right)$ for every $n \geq 6$

viii) $d\left(P_{7n-1}^3,\ n\right)\ =\ n\ +\ 1$ for every $n \in N$

Proof

i) Since $d\left(P_{7n}^3,\ n\right)\ =\ \{5,\ 12,\ 19,\ldots 7k\ -\ 2\}$

$\therefore$ we have $d\left(P_{7n}^3,\ n\right)\ = 1$

ii) Since $D\left(P_n^3,\ n\right)\ =\ \{[n]\}$; we have the result

$\therefore\ d\left(P_n^3,\ n\right)\ =\ 1$ for every $n \in N$

iii) Since $D\left(P_n^3,\ n-1\right)\ =\ \{[n]\ -\ \{x\}/x\hat{I}[n]\}$. we have the result $d\left(P_n^3,\ n-1\right)\ =\ n.$

iv) By induction on n

The result is true for $n\ =\ 3$

LHS $d\left(P_3^3,\ 1\right)\ =\ 3$

RHS $3C_2 = \frac{3\ \times\ 2}{1\ \times\ 2} = 3$

$\therefore$ LHS = RHS

$\therefore$ The result is true for $n\ =\ 3$

Now suppose that the result is true for all numbers less than 'n' and we prove it for n.

By result 3.2,

$$d(P_n^3,\ n-3)\ =\ d(P_{n-1}^3,\ n-4)\ +\ d(P_{n-2}^3,\ n-4)\ +\ d(P_{n-3}^3,\ n-4) + d(P_{n-4}^3,\ n-4) + d(P_{n-5}^3,\ n-4) + d(P_{n-6}^3,\ n-4) + d(P_{n-7}^3,\ n-4)$$

$$=\ (n\ -\ 1)C_2 +\ (n\ -\ 2)\ +\ 1$$

$$= \frac{(n-1)(n-2)}{2} + (n-2) + 1$$

$$= \frac{(n-1)(n-2)}{2} + (n-1)$$

$$= \frac{(n-1)}{2} + [(n-2)+2]$$

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$$= \frac{(n-1)n}{}$$

$$= \frac{n(n2-1)}{2}$$

$$=\ nC_2$$

$\therefore d(P_{7n}^3,\ n-2)\ =\ nC_2$ for every $n \geq 3$.

v) By induction on n

The result is true for n = 4.

LHS $d\left(P_4^3, 1\right)\ =\ 4$

RHS $4C_3 = \frac{4\ \times\ 3\ \times\ 2}{1\ \times\ 2\ \times\ 3} = 4$

$\therefore$ LHS = RHS

∴ The result is true for n = 4.

Now suppose that the result is true for all numbers less than 'n' and we prove it for 'n'

By result 3.2,

$$d(P_n^3, n-2) = d(P_{n-1}^3, n-3) + d(P_{n-2}^3, n-3) + d(P_{n-3}^3, n-3) + d(P_{n-4}^3, n-3) + d\, d(P_{n-5}^3, n-3) + d\, d(P_{n-6}^3, n-3) + d(P_{n-7}^3, n-3)$$

$$=(n-1)C_3 + (n-2)C_2 + (n-3) + 1.$$

$$= \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-2)(n-3)}{2} + (n-3)+1$$

$$= \frac{1}{6}\left[(n-1)(n-2)(n-3) + 3(n-2)(n-3) + 6(n-2)\right]$$

$$= \frac{(n-2)}{6}[(n-1)(n-3) + 3(n-3) + 6]$$

$$= \frac{(n-2)}{6}[(n-1)(n-3) + 3n - 9 + 6]$$

$$= \frac{(n-2)}{6}[(n-1)(n-3) + 3(n-1)]$$

$$= \frac{(n-2)(n-1)}{6}[(n-3+3)]$$

$$= \frac{n(n-1)(n-2)}{6}$$

$\therefore\ d(P_n^3, n-3) = nC_3$ for every $n \ge 4$.

vi) By induction on n

The result is true for n = 5.

LHS $d(P_5^3, 1) = 3$

RHS $5C_4 - 2 = \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4} - 2 = 3$

∴LHS = RHS

∴The result is true for n = 4.

Now suppose that the result is true for all numbers less than 'n' and we prove it for 'n' by result 3.2,

$$d(P_n^3, n-4) = d(P_{n-1}^3, n-5) + d(P_{n-2}^3, n-5) + d(P_{n-3}^3, n-5) + d(P_{n-4}^3, n-5) + d(P_{n-5}^3, n-5) + d(P_{n-6}^3, n-5) + d(P_{n-7}^3, n-5)$$

$$= (n-1)C_4 - 2 + (n-2)C_3 + (n-3)C_2 + (n-4) + 1.$$

$$= \frac{(n-1)(n-2)(n-3)(n-4)}{24} - 2 + \frac{(n-2)(n-3)(n-4)}{6} + \frac{(n-3)(n-4)}{2} + (n-3)$$

$$= [(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-3)] - 2$$

$$= [(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-3)] - 2$$

$$= \frac{(n-3)}{24}[(n-1)(n-2)(n-4) + 4(n-2)(n-4) + 12(n-4) + 24] - 2$$

$$= \frac{(n-3)}{24}[(n-1)(n-2)(n-4) + 4(n-2)(n-4) + 12n-48 + 24)] - 2$$

$$= \frac{(n-3)}{24}\left[(n-1)(n-2)(n-4) + 4(n-2)(n+4) + 12(n-2)\right] - 2$$

$$= \frac{(n-3)(n-2)}{24}[(n-1)(n-4) + 4(n-4) + 12] - 2$$

$$= \frac{(n-3)(n-2)}{24}[(n-1)(n-4) + 4n-16 + 12] - 2$$

$$= \frac{(n-3)(n-2)}{24}[(n-1)(n-4) + 4(n-1)] - 2$$

$$= \frac{(n-3)(n-2)(n-1)}{24}[n - 4 + 4] - 2$$

$$= \frac{(n-3)(n-2)(n-1)n}{24} - 2$$

$$= \frac{n(n-3)(n-2)(n-1)}{24} - 2$$

$$= nC_4 - 2$$

$\therefore\ d(P_n^3, n-4) = nC_4 - 2$ for every $n \ge 5$.

vii) By induction on n

The result is true for n = 6.

LHS $d(P_6^3, 1) = 2$

RHS $6C_5 - 2(6-4) = \frac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5} - 2(2) = 6 - 4 = 2.$

∴LHS = RHS

Now suppose that the result is true for all numbers less than 'n' and we prove it for n.

By result 3.2,

$$d(P_n^3, n-5) = d(P_{n-1}^3, n-6) + d(P_{n-2}^3, n-6) + d(P_{n-3}^3, n-6) + d(P_{n-4}^3, n-6) + d(P_{n-5}^3, n-6) + d(P_{n-6}^3, n-6) + d(P_{n-7}^3, n-6)$$

$$= (n-1)C_5 - 2\left[(n-1) - 4\right] + (n-2)C_4 - 2 + (n-3)C_3 + (n-4)C_2 + (n-5) + 1.$$

$$= (n-1)C_5 - 2\left[(n-1) - 4\right] + (n-2)C_4 - 2 + (n-3)C_3 + (n-4)C_2 + (n-5) + 1.$$

$$= \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \times 2 \times 3 \times 4 \times 5} - 2[(n-1-4)] + \frac{(n-2)(n-3)(n-4)(n-5)}{1 \times 2 \times 3 \times 4} - 2 + \frac{(n-3)(n-4)(n-5)}{1 \times 2 \times 3} + \frac{(n-4)(n-5)}{1 \times 2} + (n-5) + 1$$

$$= \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{120} - 2\left[(n-5)\right] + \frac{(n-2)(n-3)(n-4)(n-5)}{24} - 2 + \frac{(n-3)(n-4)(n-5)}{6} + \frac{(n-4)(n-5)}{2}n - 4$$

$$= \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{120} - 2n + 10 + \frac{(n-2)(n-3)(n-4)(n-5)}{24} - 2 + \frac{(n-3)(n-4)(n-5)}{6} + \frac{(n-4)(n-5)}{2} + (n-4)$$

$$= \Big[\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{120} + \frac{(n-2)(n-3)(n-4)(n-5)}{24} + \frac{(n-3)(n-4)(n-5)}{6} + \frac{(n-4)(n-5)}{2} + (n-4)\Big] - 2n + 10 - 2.$$

$$= \frac{1}{120}[(n-1)(n-2)(n-3)(n-4)(n-5) + 5(n-2)(n-3)(n-4)(n-5) + 20(n-3)(n-4)(n-5) + 60(n-4)(n-5) + 120(n-4)] - 2n + 8.$$

$$= \frac{(n-4)}{120}[(n-1)(n-2)(n-3)(n-5) + 5(n-2)(n-3)(n-5) + 20(n-3)(n-5) + 60(n-5) + 120] - 2(n-4)$$

$$= \frac{(n-4)}{120}[(n-1)(n-2)(n-3)(n-5) + 5(n-2)(n-3)(n-5) + 20(n-3)(n-5) + 60n - 300 + 120] - 2(n-4)$$

$$= \frac{(n-4)}{120}[(n-1)(n-2)(n-3)(n-5) + 5(n-2)(n-3)(n-5) + 20(n-3)(n-5) + 60(n-3)] - 2(n-4).$$

$$= \frac{(n-4)(n-3)}{120}[(n-1)(n-2)(n-5) + 5(n-2)(n-5) + 20(n-5) + 60] - 2(n-4).$$

$$= \frac{(n-4)(n-3)}{120}[(n-1)(n-2)(n-5) + 5(n-2)(n-5) + 20n - 100 + 60] - 2(n-4).$$

$$=\frac{(n-4)(n-3)}{120}[(n-1)(n-2)(n-5)$$

$$+5(n-2)(n-5)+20(n-2)]-2(n-4).$$

$$=\frac{(n-4)(n-3)(n-2)}{120}[(n-1)(n-5)+5(n-5)+20]-2(n-4)$$

$$=\frac{(n-4)(n-3)(n-2)}{120}[(n-1)(n-5)+5(n-1)]-2(n-4)$$

$$=\frac{(n-4)(n-3)(n-2)}{120}[(n-1)(n-5)+5n-25+20]-2(n-4)$$

$$=\frac{(n-4)(n-3)(n-2)}{120}[(n-1)(n-5)+5(n-1)]-2(n-4)$$

$$=\frac{(n-4)(n-3)(n-2)(n-1)}{120}[n-5+5]-2(n-4)$$

$$=\frac{(n-4)(n-3)(n-2)(n-1)n}{120}-2(n-4)$$

$$=\frac{n(n-1)(n-2)(n-3)(n-4)}{120}-2(n-4)$$

$\therefore d\left(P_n^3, n-5\right) = nC_5 - 2(n-4)$ for every $n \geq 6$.

viii) From the table it is true, $d\left(P^3_{n-1}, n\right) = n+1$ for every $n \in N$.

Theorem 2

$$\sum_{i=k}^{7k} d(P_i^3, k) = \sum_{i=2}^{7k-7} d(P_i^3, k-1)$$

Proof by induction on n,

First suppose that k = 2 then $\sum_{i=k}^{14} d(P_i^3, 2) = 112 = 7\sum_{i=1}^{14} d(P_i^3, i)$

$$\sum_{i=k}^{7k} d(P_i^3, k) = \sum_{i=k}^{7k} d(P_{i-1}^{\ 3}, k-1) + \sum_{i=k}^{7k} d(P_{i-2}^{\ 3}, k-1) + \sum_{i=k}^{7k} d(P_{i-3}^{\ 3}, k-1)$$

$$+\sum_{i=k}^{7k} d(P_{i-4}^{\ 3}, k-1) + \sum_{i=k}^{7k} d(P_{i-5}^{\ 3}, k-1) + \sum_{i=k}^{7k} d(P_{i-6}^{\ 3}, k-1) + \sum_{i=k}^{7k} d(P_{i-7}^{\ 3}, k-1)$$

$$\sum_{i=k-1}^{7(k-1)} d(P_{i-1}^{\ 3}, k-2) + 7\sum_{i=k-1}^{7(k-1)} d(P_{i-2}^{\ 3}, k-2) + 7\sum_{i=k-1}^{7(k-1)} d(P_{i-3}^{\ 3}, k-1)$$

$$+7\sum_{i=k-1}^{7(k-1)} d(P_{i-4}^{\ 3}, k-2) + 7\sum_{i=k-1}^{7(k-1)} d(P_{i-5}^{\ 3}, k-2)$$

$$+7\sum_{i=k-1}^{7(k-1)} d(P_{i-6}^{\ 3}, k-2) + 7\sum_{i=k-1}^{7(k-1)} d(P_{i-7}^{\ 3}, k-2)$$

$$= 7\sum_{i=K-1}^{7k-7} d(P_i^3, k-1)$$

$$\sum_{i=K}^{7k} d(P_i^3, k) = 7\sum_{i=K-1}^{7k-7} d(P_i^3, k-1)$$

$\therefore$ Hence the theorem

CONCLUSION

Using domination Polynomial, we obtain many interesting properties and theorems. This study can be expanded to other graphs also.

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