On the injectivity of an integral operator connected to Riemann hypothesis

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ABSTRACT

The equivalent formulation of the Riemann Hypothesis (RH) given by Alcantara-Bode (1993) states: RH holds if and only if the integral operator on the Hilbert space L^2 (0, 1) having the kernel function defined by the fractional part of (y/x) is injective. This formulation reduced one of the most important unsolved problems in pure mathematics to a problem whose investigation could be made by standard techniques of the applied mathematics.

The method introduced to deal with, is based on a result obtained in this paper: an operator linear, bounded, Hermitian on a separable Hilbert space strict positive definite on a dense family of including subspaces, subspaces on which the sequence of the ratios between the smallest and largest eigenvalues of the operator restrictions on the family is bounded inferior by a strict positive constant, is injective. Using a version of the generic method for integral operators on L^2 (0, 1) we proved the injectivity of the integral operator used in the equivalent formulation of the RH.

Key Words: *Approximation Subspaces, Integral Operators, Riemann Hypothesis.*

INTRODUCTION

In his paper Alcantara-Bode proved (Theorem 1 pg. 152): the Riemann Hypothesis (RH) holds if and only if the following Hilbert-Schmidt integral operator T_φ defined on L^2 (0, 1) by:

$$(T_\varphi u)(y) = \int_0^1 \left\{ \tfrac{y}{x} \right\} u(x)dx \qquad (1)$$

has its null space $N_{T_\varphi} = \{0\}$ where the kernel function is the fractional part function of the expression between brackets [1]. This integral operator has been introduced in (Beurling, 1955) for providing first equivalent formulation of RH [2], in terms of functional analysis. Beurling equivalent formulation of RH has been used by Alcantara-Bode in the second equivalent formulation of RH in terms of the injectivity of the operator in (1). The Alcantara-Bode formulation allows a different approach of the RH by using applied mathematics techniques in finding the injectivity of this integral operator.
A very captivating view on RH could be found in [3].

We provide a method for investigating the injectivity of the linear bounded operators on separable Hilbert spaces using their approximations on dense families of subspaces. In this paper the norm used is the norm induced by the inner product. A dense family of finite dimension including subspaces is an infinite collection of subspaces $\{S_n\}$, $n \geq 2$ in H, with the properties: $S_n \subset S_{n+1}$, $n \geq 1$ and $\overline{U_{i=1}^{\infty} S_i} = \mathrm{H}$.

The idea behind this method is based on the following observations.
If a linear, bounded operator T on H strict positive definite on a dense family of including subspaces has a zero, $u \in N_T$ not null, it cannot be in any subspace of the family.

If $u \in B := \{u \in H; \|u\| = 1\}$ is not in any subspace of the family, it has its orthogonal projections on the family $\{u_n\}$, $n \geq 1$, verifying $\|u_n\| \uparrow 1$ with $n \to \infty$. Because between the orthogonal projections of u and its residuum $(u - u_n)$ there exists the following relationship $\|u_n\|^2 + \|u - u_n\|^2 = 1$, there exists n_0 such that for $n \geq n_0$, $\|u_n\| > \|u - u_n\|$. Now, if C is a constant > 0, there exists n_1 such that C $\|u_n\| > \|u - u_n\|$ once $n > n_1$.

Or, as we proved, a such element $u \in B$ could not be in the null space of the operator T, if C is a inferior bound of the set of the injectivity parameters of T, $\{\mu_n\}$, $n \geq 1$, parameters that are defined when T is also Hermitian, as the ratio between smallest and largest eigenvalues of the restrictions of T on the family's subspaces.

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Indeed, it follows from Lemma 1 that if u ∈ N_T ∩ B then $\mu_n \|u_n\| \le \|u-u_n\|$ on the approximation subspaces for n ≥ n_0 (u) and so, the sequence of the injectivity parameters {μ_n} must converge to 0 in contradiction with its inferior bounding by a strict positive constant C. So, we will set accordingly our working environment in order to build the method.

Observation 1. Suppose S is a subspace of a separable Hilbert space H on which T, an linear, bounded operator is strict positive definite: $\alpha\|v\|^2 \le \langle Tv, v\rangle$ for every v ∈ S. If there exists u ∈ N_T satisfying $\|P_{|S}u\| \ne 0$ where $P_{|S}u$ is the orthogonal projection of u on S, then

$$\alpha\|P_{|S}u\| \le \omega\|u-P_{|S}u\| \tag{2}$$

where ω > 0 verifies $\|T^*v\| \le \omega\|v\|$ for every v in S.

Proof. If u ∈ N_T then u ∈ H\S and $\|u-P_{|S}u\| \ne 0.$ If exists w ∈ $S^\perp$ such that Tw is not null and $\langle Tw, v\rangle \ne 0$, for every v not null ∈ S, then $|\langle Tw, v\rangle| \le \|w\|\|T^*v\|$. Taking $v = P_{|S}u$ and w = (v - u) follows

$$\alpha\|P_{|S}u\|^2 \le \langle T(P_{|S}u), P_{|S}u\rangle = \langle T(P_{|S}u-u), P_{|S}u\rangle \le \|u-P_{|S}u\|\|T^*(P_{|S}u)\|$$

proving the observation.

Let F = {S_n, n ≥ 1} a dense family of finite dimension including subspaces and denote L_F the class of the linear, bounded operators on H that are strict positive definite on every S_n ∈*F*. If T ∈ L_F for every v_n ∈ S_n, n ≥ 1 and for every S_n ∈*F* holds

$$\langle Tv_n, v_n\rangle \ge \alpha_n\|v_n\|^2, \qquad \alpha_n > 0 \tag{3}$$

Applying the Observation 1 on the members of the dense family *F* follows:

Lemma 1. Let T∈ L_F. Suppose u ∈ N_T and denote $\beta_n(u) := \|u-u_n\|$ with $u_n = P_{|S_n}u$, the orthogonal projection of u on S_n. Then

$$\mu_n\|u_n\| \le \beta_n(u), \qquad n \ge n_0 := n_0(u) \tag{4}$$

where $\mu_n = \alpha_n/\omega_n, \omega_n$ verifying $\|T^*v\| \le \omega_n\|v\|$ for every v ∈ S_n, $n_0(u)$ being the index of the subspace from where the projections u_n are not null.

Proof. A such index $n_0(u)$ there exists for every u because of the density of the family *F* in H adding, for n < n_0 (4) is trivial for every u and any subspace once the orthogonal projections are zero for n < n_0. The inequality follows applying the Observation 1 on each subspace S_n from the dense family.

A criteria for linear operators injectivity

Let T ∈ L_F. If u∈ N_T then $u \notin S_n$, n ≥ 1, following $N_T \subset H \setminus \cup_{n=1}^{\infty} S_n$.

We define the set of the eligible normalized zeros of the linear bounded strict positive operators on the family *F* by:

$$B_0 := B_0(F) = \{u; u \in (H \setminus \cup_{n=1}^{\infty} S_n), \qquad \|u\| = 1\}$$

So, if u∈ N_T then $u/\|u\| \in N_T \cap B_0$.

On the eligible set B_0 we consider the following expression:

$$\theta_n(u) := \theta_n^T(u) = \beta_n^2(u) - \mu_n^2\|u_n\|^2 = 1-(\mu_n^2+1)\|u_n\|^2$$

where we exploited the relationship for an eligible u between its orthogonal projection on S_n and its residuum (u - u_n), $\|u\|^2 = \|u_n\|^2 + \beta_n^2(u)$.

Theorem 1 (Injectivity Criteria). Let T∈ L_F. If there exists a strict positive constant C independent of n, such that $\mu_n(T) \ge C > 0$ for every n ≥ 1, then T is injective.

Proof. Suppose that the sequence {μ_n}, n ≥ 1 is inferior bounded by a constant C independent of n, strict positive and let u ∈ B_0. If there exists an infinite subsequence of subspaces from the family for which $\theta_{n_m}(u) \ge 0$, $n_m \ge n_0(u)$ then: $\|u_{n_m}\|^2 \le 1/(1+\mu_{n_m}^2) \le 1/(1+C^2) < 1$ and $\lim_{m\to\infty}\|u_{n_m}\|^2 \le 1/(1+C^2) < 1$

Or, $\{u_{n_m}\}$, $n_m \ge n_0(u)$, like {u_n}, n ≥ 1, should converge in norm to 1 on the approximation subspaces for every u ∈ B_0.

Thus, the inequality $\theta_{n_m}(u) \ge 0$ could not take place on an infinity of subspaces of the family. We should have instead at most only a finite number of subspaces verifying it.

If there exists a finite number of subspaces on which $\theta_{n_m}(u) \ge 0$, $n_m \ge n_0(u)$ then there exists an index $n_1(u)$ such that we are in the situation $\theta_n(u) < 0$ for every n ≥ $n_1(u)$.

If $\theta_n(u) < 0$ holds for n ≥ $n_1(u)$, then $\beta_n(u) < \mu_n\|u_n\|, n \ge n_1(u).$

But, u satisfying such relationship could not be in N_T because a reverse inequality given by Lemma 1 holds for eligible u ∈ N_T. The only restriction we put on u ∈ H, has been its eligibility, u ∈ B_0. Thus, if {μ_n}, n ≥ 1 is inferior bounded, does not exists u ∈ B_0 that is in N_T. Or, B_0 contains all normalized zeros of T, so N_T ={0} Q.E.D.

Lemma 2. If T is linear, bounded strict positive definite on F and the sequence of the injectivity parameters are bounded by a constant, then T is injective. This property is a immediate consequences of the property of boundness of T on H and so on every subspace of the family.

If T is not strict positive definite on *F* then we consider its associated Hermitian (T*T). If (T*T) is not strict positive definite on *F* then it has a zero in one of subspaces of *F* and so, T and (T*T) are not injective. While the strict positivity of the operator or of its Hermitian is mandatory for injectivity, the injectivity criteria Theorem 1 is only a sufficient condition as we observed from our example on the last paragraph.

Nowhere is included the Hermitian property in the proof of the Injectivity Criteria. We will replace the operator with its Hermitian when the original operator is not strict positive on the dense family of approximation subspaces or when we do not have enough information about its positivity - the reason is, both have the same null space and the Hermitian is positive definite on H.

When the linear operator T is Hermitian and strict positive on a dense family of finite dimension subspaces, then $\mu_n = \lambda_{min}(T_n)/\lambda_{max}(T_n)$. For showing it, it is enough to consider the finite dimension (sub)space S_n equipped with the orthonormal basis of the eigenfunctions of T_n in order to express α_n and ω_n function of the eigenvalues of T_n.

Approximations of the Integral Operators

Let consider in the separable Hilbert H:= L^2 (0, 1) the dense family of including finite dimension approximation subspaces $F_\chi := \{S_h;\ nh = 1, n \geq 2\}$, where $S_h := span\{\chi_{h,k}, k = 1,n, nh = 1\}$ spanned by indicator functions of disjoint intervals covering the domain (0,1):

$$\chi_{h,k} := \chi_{h,k}(t) = \begin{cases} 1 & t \in \Delta_k := ((k-1)h, kh] \\ 0 & otherwise \end{cases} \qquad (5)$$

This family of approximation subspaces has been used in dealing with decay rate of convergence to zero of the integral operators' eigenvalues, operators having the kernel functions like Mercer kernels [4], i.e. Hermitian and at least continue [5-7]. We will use this family for investigating the injectivity of the linear bounded integral operators. The family of functions $\{\chi_{h,k}\}$, k = 1,n, nh =1, defines a trace class integral operator with the kernel function $r_h(x,y) = h^{-1}\sum_{k=1}^{n}\chi_{h,k}(x)\chi_{h,k}(y)$ that is an orthogonal projection in L^2 (0,1) on S_h having the orthogonal eigenfunctions $\{\chi_{h,k}\}, k = 1,n$ (see [5]). In fact, each $S_h \in F_\chi$ is generated by the families of the orthogonal eigenfunctions of the projection operator $P_h := P_{|S_h}$. Thus, the the kernel function φ of the linear integral operator

$$T_\varphi u := (T_\varphi u)(y) = \int_0^1 \varphi(y,x)u(x)dx \qquad (6)$$

has the discrete approximations on S_h, nh = 1, n ≥ 2 given by [5]:

$$\varphi_h(y,x) = h^{-1}\sum_{k=1}^{n}\chi_{h,k}(y)\varphi(y,x)\chi_{h,k}(x) \qquad nh = 1, n \geq 2 \qquad (7)$$

Observation 2. Taking a look at the kernel function φ_h, we observe that it is a sum of 'orthogonal piecewise functions' once each term in the sum, $\varphi_h^k(y,x) = \chi_{h,k}(y)\varphi(y,x)\chi_{h,k}(x) \quad k = 1,n, \ \ nh = 1$ is zero valued outside the square $(\Delta_k \times \Delta_k)$. So, the matrix representation of the corresponding restriction operator will be 1-diagonal sparse matrix.

The dense family has including finite dimensional subspaces, $S_h \subset S_{h/2}$, so, if the integral operators T_{φ_h} having the kernel functions φ_h are strict positive definite then we fulfill the hypothesis of the Injectivity Criteria for the linear, bounded integral operator in (6). Thus we could proceed computing the components α_h and ω_h of their injectivity parameters $\{\mu_h\}$ (see, (4)) using the matrix representation of the restrictions of the operator T_φ on the finite dimension subspaces $S_h \in F_\chi$.

- A formula for α_h. Let $v_h = \sum_{k=1,n} c_k \chi_{h,k} \in S_h$.

Then $\langle T_\varphi v_h, v_h \rangle = \sum_{k=1,n} c_k \overline{c_k} d_{kk}^h = c_h^T M_{\varphi_h} \bar{c}_h$ where $M_{\varphi h}$ is the matrix representation of the restriction operator T_{φ_h} to S_h having the entries

$$d_{kk}^h = \langle T_\varphi \chi_{h,k}, \chi_{h,k} \rangle = h^{-1}\int_{\Delta_k}\int_{\Delta_k}\chi_{h,k}(y)\varphi(y,x)\chi_{h,k}(x)dxdy$$

Obviously, M_{φ_h} is a diagonal matrix due to $\chi_{h,i}\chi_{h,j} = \delta_{ij}, i, j = 1, n.$

If its entries are positive valued, then $\langle T_\varphi v_h, v_h \rangle \geq h^{-2}\min_{k=1,n}(d_{kk}^h)\|v_h\|^2$ and

$$\alpha_h = h^{-2}\min_{k=1,n}\int_{\Delta_k}\int_{\Delta_k}\chi_{h,k}(y)\varphi(y,x)\chi_{h,k}(x)dxdy \qquad (8)$$

- A formula for ω_h. From

$$\|T_\varphi^* v_h\|^2 = \sum_{k=1,n} c_k \overline{c_k} d_{kk}^{h*} \leq h^{-1}\max_{k=1,n}(d_{kk}^{h*})\|v_h\|^2$$ where

$$d_{kk}^{h*} = \langle T_\varphi^*\chi_{h,k}, T_\varphi^*\chi_{h,k} \rangle = \|T_\varphi^*\chi_{h,k}\|^2$$

$$= h^{-2}\int_{\Delta_k}\left[\left(\int_{\Delta_k}\chi_{h,k}(y)\varphi(x,y)\chi_{h,k}(x)dx\right)\left(\int_{\Delta_k}\overline{\chi_{h,k}(y)\varphi(x,y)\chi_{h,k}(x)}dx\right)\right]dy$$

we obtain $\|T_\varphi^* v_h\| \leq h^{-1/2}\max_{k=1,n}(d_{kk}^{h*})^{1/2}\|v_h\|$. Thus

$$\omega_h = h^{-3/2}\max_{k=1,n}\sqrt{\omega_{h,k}} \qquad (9)$$

Where

$$\omega_{h,k} = \int_{\Delta_k}\left[\left(\int_{\Delta_k}\chi_{h,k}(y)\varphi(x,y)\chi_{h,k}(x)dx\right)\left(\int_{\Delta_k}\overline{\chi_{h,k}(y)\varphi(x,y)\chi_{h,k}(x)dx}\right)\right]dy.$$

On Alcantara-Bode Hypothesis

The kernel function of the linear, bounded Hilbert-Schmidt integral operator in (1), $\varphi(y, x) = \{y/x\}$ as a function $\in L^2(0, 1)^2$ is almost continue everywhere because its discontinuities lie on lines in $(0,1)^2$ of the form y=kx, k ∈N, that is a countable set of Lebesgue measure zero subsets and so, of Lebesgue measure zero and so will be the kernel of its associated Hermitian integral operator. In line with [5-6] we will consider the dense family of approximation subspaces F_χ introduced in previous paragraph in order to investigate its injectivity. The entries of the diagonal matrix M_{φ_h}, the representation of the restriction of the operator T_φ on S_h defined by:

$d_{kk}^h = \langle T_\varphi\chi_{h,k}, \chi_{h,k} \rangle = \int_{\Delta_k}\int_{\Delta_k}\left\{\frac{y}{x}\right\}dxdy,\ nh = 1,\ n \geq 2.$ are:

$$d_{11}^h = \frac{h^2}{4}(3-2\gamma); d_{kk}^h = \frac{h^2}{2}\left(-1+\frac{2k-1}{k-1}\ln\left(\frac{k}{k-1}\right)^{k-1}\right)$$

where γ is the Euler-Mascheroni constant (~= 0.5772156). The sequence $\left(-1+\frac{2k-1}{k-1}\ln\left(\frac{k}{k-1}\right)^{k-1}\right)$ is monotone and converges to 0.5 for k → ∞. These entries are strict positive valued, bounded by $h^{-2}d_{11}^h$ and $h^{-2}d_{22}^h$ on S_h, showing that $T_\varphi \in F_\chi$ having the strict positivity parameters given by

$$\alpha_h = h^{-2}d_{11}^h = \frac{1}{4}(3-2\gamma) \qquad (10)$$

So, α_h is a strict positive constant ∀n ≥ 2, nh = 1. We could apply Lemma 2 to obtain the injectivity without computing the ω_h parameters knowing that T_φ^* is bounded. However, we will proceed with the computation of ω_h (See the note from the next paragraph).
Taking in consideration that the kernel function of the adjoint operator is $\overline{\varphi(x, y)} = \varphi(x, y)$ verifies sup $(\{x/y\})<1$ on each square $(\Delta_k \times \Delta_k)$ of area h^2, nh = 1, n ≥ 2, the integrals in $\omega_{h,k}$ are superior bounded by h, obtaining from (9) $\omega_h < 1$ and so, $\mu_h \geq \alpha_h = \frac{1}{4}(3\text{-}2\gamma)$. Then, T_φ is injective due to the injectivity criteria.

Has no meaning to use the associated Hermitian operator once the operator is strict positive on F_χ and injective, both having the same null space.
Anyway, let's consider the associated Hermitian operator $T_\psi := T_\varphi^* T_\varphi$ of the integral operator in (1) having the kernel function $\psi(y,x) = \int_0^1 \varphi(y,t)\varphi(x,t)dt$

It is easy to observe that $\alpha_h(T_\psi) = \alpha_h^2(T_\varphi) = \frac{1}{16}(3-2\gamma)^2, nh = 1, n \geq 2.$
So, $T_\psi \in L_{F_\chi}$ is injective if we apply Lemma 2. On the other hand, $\left\|(T_\varphi^* T_\varphi)_h v_h\right\| \leq \left\|(T_\varphi^* T_\varphi)_h\right\| \|v_h\|$ shows that $\omega_h = \left\|(T_\varphi^* T_\varphi)_h\right\| \leq \left\|(T_\varphi^* T_\varphi)\right\|$ is bounded by a constant as is α_h. Follows $\mu_h\left((T^*T)_h\right)$ is inferior bounded by a constant showing the injectivity of the operator avoiding Lemma 2. We just proved using the Injectivity Criteria:

Theorem 2. The linear bounded Hilbert-Schmidt integral operator T_φ on L^2 (0,1) having the kernel function the fractional part function $\varphi(y,x) := \{\frac{y}{x}\}$ defined by

$$(T_\varphi u)(y) := \int_0^1 \varphi(y,x)u(x)dx \quad (11)$$

used for providing the equivalent formulation of Riemann Hypothesis [1], is injective, or equivalently, has the null space $N_{T_\varphi} = \{0\}$.

Some numerical aspects of the method
The method of operator projections using the finite rank operators associated to the family F_χ is backing up our choice to use the dense family for approximating the Beurling - Alcantara-Bode integral operator and we consider a consistent discretization approach. To justify the affirmation, let φ a kernel function on $L^2(0, 1)^2$ and its approximations on F_χ be

$$\varphi_h(x,y) = h^{-t} \sum_{k=1,n} \chi_{h,k}(y)\varphi(y,x)\chi_{h,k}(x), nh = 1, \quad t \in N \cup \{0\}. \quad (12)$$

We observe from (8) and (9) that the injectivity parameters are independent of t by replacing h^{-1} with h^{-t}. We find the kernel approximations for t = 0 in [6]. For t = 1 we find it in [5]. We have: $\mu_h^{(t=0)} = \mu_h^{(t=1)}$.
The injectivity criteria could be reformulated: if the components of the injectivity parameters alpha and omega are of the same order of h, then the linear operator is injective. This is the reason why Lemma 2 could be applied with caution choosing a consistent discretisation using orthogonal projection operators on approximation subspaces like in [5].

Observation 3. The following operator is injective having its injectivity parameters converging to 0.
The operator obtained as a product of identity by a power of x, I_s, s ∈ N on $L^2(0,1)$ as follows: for u ∈ L^2 (0,1)

$$(I_s u)(x) = x^s u(x) \quad i.e. \quad I_s : u(x) \to x^s u(x) \quad (13)$$

I_s is linear, bounded, strict positive definite and Hermitian on $L^2(0, 1)$ and so, strict positive on F_χ. The injectivity and positivity properties on H are coming from $\langle I_s u, u\rangle = \int_0^1 w(x)\overline{w(x)}dx > 0$ for $u \neq 0$ where $w(x) = x^{s/2}u(x)$.
The components of the injectivity parameters using the theory exposed in the previous paragraphs are obtained as follows: given

$$v_h = \sum_{k=1,n} c_k \chi_{h,k} \in S_h,$$

$$\langle T_s v_h, v_h\rangle = \sum_{k=1,n} c_k \overline{c_k} \int_{\Delta_k} \chi_{h,k}(x) x^s \chi_{h,k}(x)dx$$

$\geq h^{-1}\|v_h\|^2 \min_{k=1,n} \frac{x^{s+1}}{s+1}\Big|_{(k-1)h}^{kh}$, following

$\alpha_h(T_S) = \frac{h^s}{s+1}\min_{k=1,n}(\gamma_{k,s+1})$, where $\gamma_{k,q} = k^q - (k-1)^q$.

From

$$\|T_s^* v_h\|^2 = \sum_{k=1,n} c_k \overline{c_k} \int_{\Delta_k} \chi_{h,k}(x) x^{2s} \chi_{h,k}(x)dx \leq \|v_h\|^2 \frac{h^{2s}}{(2s+1)} \max_{k=1,n} \gamma_{k,2s+1}$$

we have $\omega_h^s = \frac{h^s}{\sqrt{2s+1}} \max_{k=1,n} \sqrt{\gamma_{k,2s+1}}$. Now, $\gamma_{k,2s+1} \leq n^{2s+1} - (n-1)^{2s+1} \leq (2s+1)n^{2s}$. Then $\omega_h^s \leq \sqrt{h^{2s}n^{2s}} = 1$.
showing that for $s \geq 1$ $\alpha_h^s \to 0$ and ω_h^s is a constant.

For s=0, $T_s \equiv I$, and $\alpha_h^{s=0} = 1/2, \omega_h^{s=0} = 1$, and $\mu_h(I)$ is constant.
Instead, for $s \geq 1$ ω_h is bounded by a constant and, $\alpha_h \to 0$ together with μ_h.

Thus: T_s, $s \geq 1$, has its injectivity parameters violating the request of the injectivity criteria $(\mu_h \geq C)$ showing that the injectivity criteria Theorem 1 is only a sufficient condition for the injectivity [8].

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